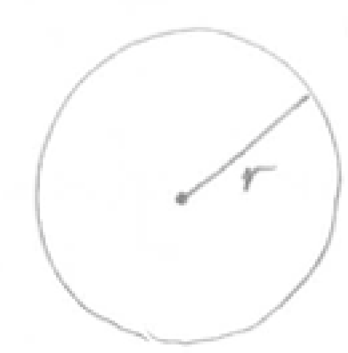
**Math 150: Quiz 2** **Name:** ______________________

*Instructions: You must show your work on all problems except problem 2. If you write only the answer, you will get zero credit for that problem.*

**1. (10 points)** The radius of a circle increases at a constant rate of 10 m/s. How fast is the area of the circle increasing when the radius is 200 m? (Note: the area of a circle of radius $r$ is $\pi r^2$.)

Let $A$ be the area

Let $r$ be its radius




Given: $\frac{dr}{dt} = 10$, $A = \pi r^2$

Want to find: $\left.\frac{dA}{dt}\right|_{r=200}$

$$A = \pi r^2$$

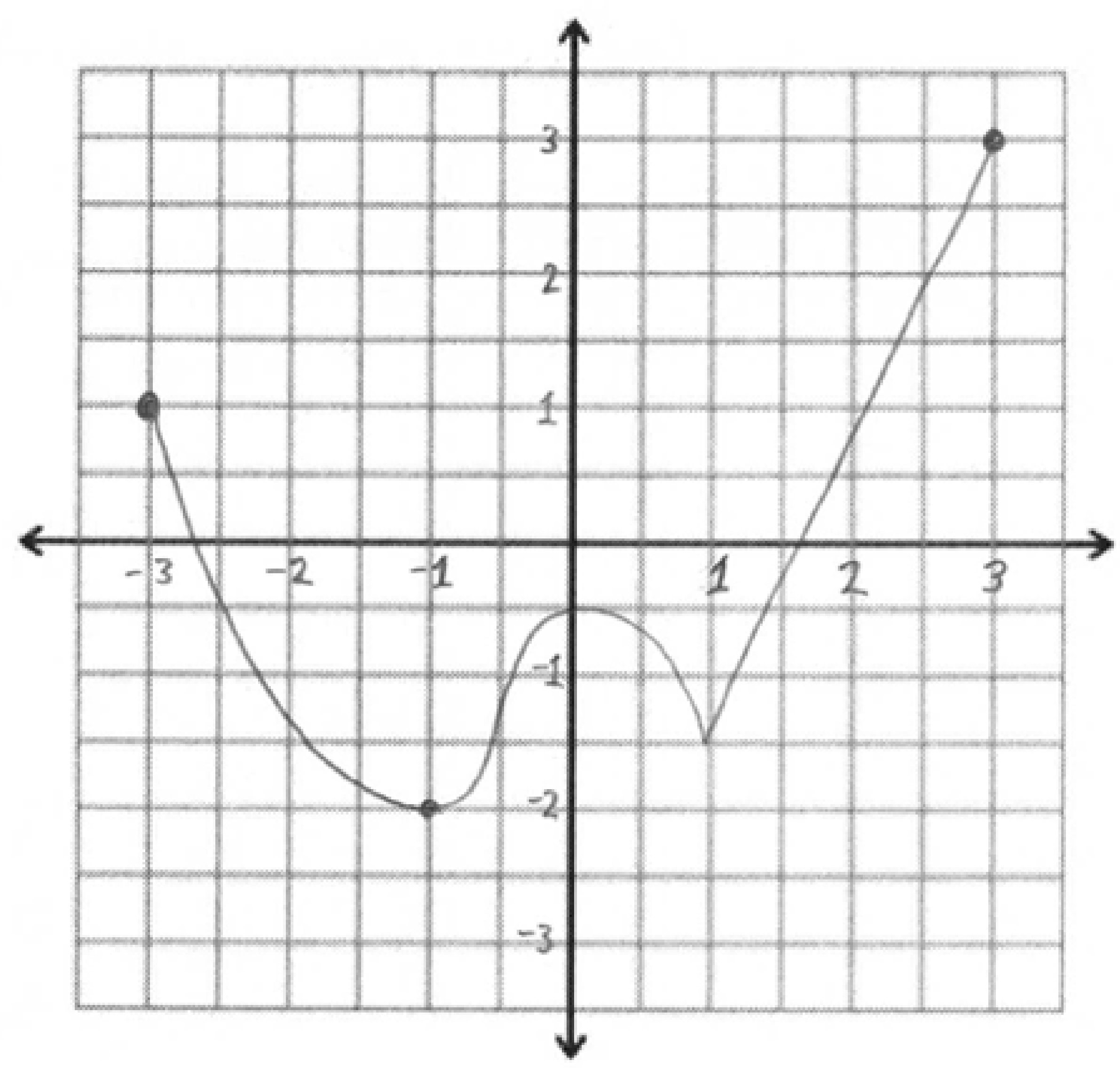
$$\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$$

$$\frac{dA}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt}$$

$$\left.\frac{dA}{dt}\right|_{r=200} = \pi \cdot 2 \cdot 200 \cdot 10 = 4000\pi \ m^2/s$$

**2. (10 points)** Sketch a graph of a single continuous function $f$ such that

- $f$ has domain $[-3, 3]$
- $f(-1) = -2$
- $f$ has an absolute maximum at $x = 3$
- $f$ has an absolute minimum at $x = -1$
- $f$ has local minimums at $x = -1$ and $x = 1$
- $f$ is *not* differentiable at $x = 1$
- $f$ has a local maximum at $x = 0$



**3. (10 points)** Find the absolute maximum and absolute minimum values of $f(x) = (x^2 - 1)^3$ on the interval $[-1, 2]$.

$$f(x) = (x^2-1)^3$$
$$f'(x) = 3(x^2-1)^2(2x)$$
$$= 6x(x^2-1)^2$$

So $f'(x) = 0$ iff $x = 0$ or $x = \pm 1$

$$\left.\begin{aligned} f(-1) &= 0 \\ f(0) &= -1 \\ f(1) &= 0 \\ f(2) &= 3^3 = 27 \end{aligned}\right\}$$

By the Closed Interval Method, the abs. max value of $f$ on $[-1,2]$ is $27$ and the abs min value is $-1$.

**4. (10 points)** Let $f(x) = 5x^3 - 3x^5$. Find the intervals on which $f$ is increasing or decreasing and the intervals where $f$ is concave up or concave down. Find the local maximum and local minimum values of $f$ and the inflection points.

$f'(x) = 15x^2 - 15x^4 = 15x^2(1-x^2)$

$f'(x) = 0$ iff $x = 0$ or $x = \pm 1$

$f'(x)$: $-$ | $-1$ | $+$ | $0$ | $+$ | $1$ | $-$

sample points:
$f(-2) = 15 \cdot 4 \cdot (1-4) < 0$
$f(-\frac{1}{2}) = 15 \cdot \frac{1}{4} \cdot (1-\frac{1}{4}) > 0$
$f(\frac{1}{2}) = 15 \cdot \frac{1}{4} \cdot (1-\frac{1}{4}) > 0$
$f(2) = 15 \cdot 4 \cdot (1-4) < 0$

So $f$ is decreasing on $(-\infty,-1) \cup (1,\infty)$ and increasing on $(-1,1)$.
By the first derivative test, $f$ has a local min at $x=-1$ and a local max at $x=1$.

$f''(x) = 30x - 60x^3 = 30x(1-2x^2)$

$f''(x) = 0$ iff $x = 0$ or $1 - 2x^2 = 0$
iff $x = 0$ or $\frac{1}{2} = x^2$
iff $x = 0$ or $x = \pm\sqrt{\frac{1}{2}}$

$f''(x)$: $+$ | $-\sqrt{\frac{1}{2}}$ | $-$ | $0$ | $+$ | $\sqrt{\frac{1}{2}}$ | $-$

Sample points:
$f(-1) = 30(-1)(1-2\cdot 1) > 0$
$f(-\frac{1}{2}) = 30(-\frac{1}{2})(1-2\cdot\frac{1}{4}) < 0$
$f(\frac{1}{2}) = 30(\frac{1}{2})(1-2\cdot\frac{1}{4}) > 0$
$f(1) = 30(1)(1-2\cdot 1) < 0$

NOTE:
Why is $1 > \sqrt{\frac{1}{2}}$?
Since $1 > \frac{1}{2}$, it follows $1 = \sqrt{1} > \sqrt{\frac{1}{2}}$.

Why is $\frac{1}{2} < \sqrt{\frac{1}{2}}$?
Since $\frac{1}{4} < \frac{1}{2}$, it follows $\frac{1}{2} = \sqrt{\frac{1}{4}} < \sqrt{\frac{1}{2}}$.

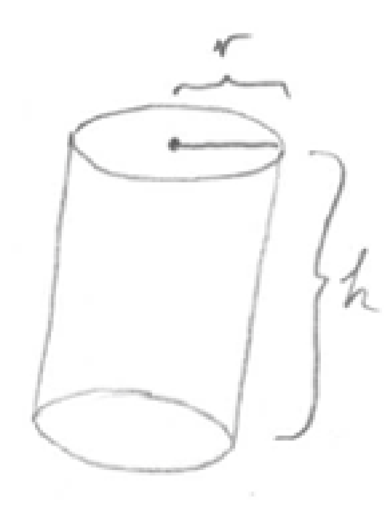
So $f$ is concave up on $(-\infty, -\sqrt{\frac{1}{2}}) \cup (0, \sqrt{\frac{1}{2}})$ and concave down on $(-\sqrt{\frac{1}{2}}, 0) \cup (\sqrt{\frac{1}{2}}, \infty)$. Also, $f$ has inflection points $\left(-\frac{1}{\sqrt{2}}, -\frac{5}{2}\frac{1}{\sqrt{2}} + \frac{3}{4}\frac{1}{\sqrt{2}}\right)$,
$(0,0)$, and
$\left(\frac{1}{\sqrt{2}}, \frac{5}{2}\frac{1}{\sqrt{2}} - \frac{3}{4}\frac{1}{\sqrt{2}}\right)$.

**5. (10 points)** You are designing a cylindrical paper drinking cup with a capacity of 10 cm$^3$. Find the height and radius of the cup that will use the least amount of paper. *Be sure to check that the local minimum you find is in fact the absolute minimum (hint: consider the concavity).* (Note: the volume of a cylinder is $\pi r^2 h$ and the surface area of a cylinder including the base but not the top is $2\pi rh + \pi r^2$.)

Given: $V = \pi r^2 h$

$V = 10$

We want to maximize: $A = 2\pi rh + \pi r^2$




$$10 = \pi r^2 h \Rightarrow h = \frac{10}{\pi r^2}$$

$$A = 2\pi rh + \pi r^2$$

$$A(r) = 2\pi r\left(\frac{10}{\pi r^2}\right) + \pi r^2$$

$$= 20r^{-1} + \pi r^2$$

$$A'(r) = -20r^{-2} + 2\pi r$$

$$A'(r) = 0 \text{ iff } 20r^{-2} = 2\pi r$$

$$10 = \pi r^3$$

$$\sqrt[3]{\frac{10}{\pi}} = r$$

Note: $A''(r) = 40r^{-3} + 2\pi$, so $A''(r) > 0$ for all $r > 0$.

Hence $r = \sqrt[3]{\frac{10}{\pi}}$ is not only a local min but also a global min.

So $A$ is minimized when $r = \sqrt[3]{\frac{10}{\pi}}$ and $h = \frac{10}{\pi r^2} = \frac{10}{\pi\left(\sqrt[3]{\frac{10}{\pi}}\right)^2} = \frac{10}{\pi}\left(\frac{10}{\pi}\right)^{-2/3} = \sqrt[3]{\frac{10}{\pi}}$.