Notes on ε-δ proofs

Based on notes by Barry Cherkas — Hunter Math 156 – Natoli

1 Convergence and divergence of sequences

Definition 1. A **sequence** of real numbers is an ordered list $\langle a_1, a_2, a_3, \ldots \rangle$ where $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$. The sequence can also be denoted by $\langle a_n \rangle$.

Definition 2. A sequence $\langle a_n \rangle$ **converges to** $L \in \mathbb{R}$ if for any real number $\varepsilon > 0$, there is some $N_\varepsilon \in \mathbb{N}$ such that whenever $n > N_\varepsilon$, we have $|a_n - L| < \varepsilon$. This is denoted $\lim_{n\to\infty} a_n = L$ or "$a_n \to L$ as $n \to \infty$". A sequence is **convergent** if it converges to L for some $L \in \mathbb{R}$.

Note that for any ε we are given, we get to choose a new N_ε. The subscript "ε" on N_ε emphasizes this dependence. You can think of N_ε as a function of ε.

The idea behind this definition is as follows: Consider $|a_n - L|$ to be the error in approximating L by the nth element of the sequence. You might want to bound the maximum possible error to ensure that our approximation isn't that bad. Call this bound ε. Ideally, no matter how small a bound we choose, we can always go far enough out into the sequence (that is, farther than N_ε), and as long as we are far enough, our errors $|a_n - L|$ will all be smaller than ε. In this ideal situation, we say the sequence converges to L.

Definition 3. The **ceiling** $\lceil x \rceil$ of $x \in \mathbb{R}$ is the smallest integer greater than or equal to x. The **floor** $\lfloor x \rfloor$ of $x \in \mathbb{R}$ is the largest integer less than or equal to x.

E.g., $\left\lceil \frac{3}{2} \right\rceil = 2 \qquad \left\lceil -\frac{3}{2} \right\rceil = -1 \qquad \lfloor \pi \rfloor = 3 \qquad \lfloor -\pi \rfloor = -4.$

1.1 Proving convergence

To prove a sequence $\langle a_n \rangle$ converges to L, first do some preliminary scratchwork to determine what N_ε should look like given ε, then write a direct proof.

Example. Prove that $\langle a_n \rangle = \left\langle \frac{1}{n^2} + 1 \right\rangle$ converges to $L = 1$.

Preliminary scratchwork: Note that

$$|a_n - L| = \left| \left(\frac{1}{n^2} + 1 \right) - 1 \right| = \left| \frac{1}{n^2} \right| = \frac{1}{n^2}.$$

We want $|a_n - L| < \varepsilon$ for all $n > N_\varepsilon$, i.e., we want $\frac{1}{n^2} < \varepsilon$ for all $n > N_\varepsilon$. Observe that

$$\frac{1}{n^2} < \varepsilon \quad \Longleftrightarrow \quad n^2 > \frac{1}{\varepsilon} \quad \Longleftrightarrow \quad n > \sqrt{\frac{1}{\varepsilon}}.$$

Therefore, if $N_\varepsilon > \sqrt{\frac{1}{\varepsilon}}$, then $n > N_\varepsilon$ implies $n > \sqrt{\frac{1}{\varepsilon}}$, which in turn implies $\frac{1}{n^2} < \varepsilon$. We need N_ε to be a natural number, so we choose $N_\varepsilon = \left\lceil \sqrt{\frac{1}{\varepsilon}} \right\rceil$.

Proof: Suppose $\varepsilon > 0$. Let $N_\varepsilon = \left\lceil \sqrt{\frac{1}{\varepsilon}} \right\rceil$. If $n > N_\varepsilon$, then $n > \sqrt{\frac{1}{\varepsilon}}$, so $\frac{1}{n^2} < \varepsilon$. Therefore,

$$|a_n - L| = \left| \left(\frac{1}{n^2} + 1 \right) - 1 \right| = \frac{1}{n^2} < \varepsilon.$$

Example. Prove that the sequence $\langle a_n \rangle = \left\langle \frac{2n^2}{4n^2+1} \right\rangle$ converges to $\frac{1}{2}$.

Preliminary scratchwork: First we calculate $|a_n - L|$:

$$\begin{aligned} |a_n - L| &= \left| \frac{2n^2}{4n^2+1} - \frac{1}{2} \right| \\ &= \left| \frac{2(2n^2) - (4n^2+1)}{2(4n^2+1)} \right| \\ &= \left| \frac{4n^2 - 4n^2 - 1}{8n^2+2} \right| \\ &= \left| \frac{-1}{8n^2+2} \right| \\ &= \frac{1}{8n^2+2} \qquad \text{(since } 8n^2+2 \text{ is always positive).} \end{aligned}$$

We want $|a_n - L| < \varepsilon$ for all $n > N_\varepsilon$, i.e., we want $\frac{1}{8n^2+2} < \varepsilon$ for all $n > N_\varepsilon$. Now we solve this inequality for n:

$$\frac{1}{8n^2+2} < \varepsilon \iff 8n^2 + 2 > \frac{1}{\varepsilon} \iff n^2 > \frac{1}{8}\left(\frac{1}{\varepsilon} - 2\right) \iff n > \sqrt{\frac{1}{8\varepsilon} - \frac{1}{4}}.$$

There's a minor caveat: To ensure that we're always talking about real numbers, we need to avoid taking the square root of a negative number. Depending on what ε is chosen, $\frac{1}{8\varepsilon} - \frac{1}{4}$ might be negative. Let's avoid this problem by taking the absolute value before taking the square root, i.e., we want

$$n > \sqrt{\left|\frac{1}{8\varepsilon} - \frac{1}{4}\right|}.$$

Now we need to choose N_ε such that if $n > N_\varepsilon$, then $n > \sqrt{\left|\frac{1}{8\varepsilon} - \frac{1}{4}\right|}$. We can choose $N_\varepsilon = \left\lceil \sqrt{\left|\frac{1}{8\varepsilon} - \frac{1}{4}\right|} \right\rceil$.

Proof: Suppose $\varepsilon > 0$. Let $N_\varepsilon = \left\lceil \sqrt{\left|\frac{1}{8\varepsilon} - \frac{1}{4}\right|} \right\rceil$. If $n > N_\varepsilon$, then

$$\begin{aligned} n > \sqrt{\left|\frac{1}{8\varepsilon} - \frac{1}{4}\right|} &\implies n^2 > \left|\frac{1}{8\varepsilon} - \frac{1}{4}\right| > \frac{1}{8\varepsilon} - \frac{1}{4} \\ &\implies 8n^2 > \frac{1}{\varepsilon} - 2 \\ &\implies \frac{1}{8n^2+2} < \varepsilon. \end{aligned}$$

Therefore, whenever $n > N_\varepsilon$, we have

$$|a_n - L| = \left| \frac{2n^2}{4n^2+1} - \frac{1}{2} \right| = \frac{1}{8n^2+2} < \varepsilon.$$

1.2 Proving divergence

Definition 4. A sequence is **divergent** if it is not convergent.

Let's translate the definition of convergence to symbolic logic and then negate it to get a more useful definition of divergence. In symbolic logic, the sequence $\langle a_n \rangle$ is convergent if

$$\exists L \in \mathbb{R} \quad \forall \varepsilon > 0 \quad \exists N_\varepsilon \in \mathbb{N} \quad \forall n > N_\varepsilon \quad |a_n - L| < \varepsilon.$$

Its negation is

$$\begin{aligned} &\sim(\exists L \in \mathbb{R} \quad \forall \varepsilon > 0 \quad \exists N_\varepsilon \in \mathbb{N} \quad \forall n > N_\varepsilon \quad |a_n - L| < \varepsilon) \\ =\ &\forall L \in \mathbb{R} \quad \sim(\forall \varepsilon > 0 \quad \exists N_\varepsilon \in \mathbb{N} \quad \forall n > N_\varepsilon \quad |a_n - L| < \varepsilon) \\ =\ &\forall L \in \mathbb{R} \quad \exists \varepsilon > 0 \quad \sim(\exists N_\varepsilon \in \mathbb{N} \quad \forall n > N_\varepsilon \quad |a_n - L| < \varepsilon) \\ =\ &\forall L \in \mathbb{R} \quad \exists \varepsilon > 0 \quad \forall N_\varepsilon \in \mathbb{N} \quad \sim(\forall n > N_\varepsilon \quad |a_n - L| < \varepsilon) \\ =\ &\forall L \in \mathbb{R} \quad \exists \varepsilon > 0 \quad \forall N_\varepsilon \in \mathbb{N} \quad \exists n > N_\varepsilon \quad \sim(|a_n - L| < \varepsilon) \\ =\ &\forall L \in \mathbb{R} \quad \exists \varepsilon > 0 \quad \forall N_\varepsilon \in \mathbb{N} \quad \exists n > N_\varepsilon \quad |a_n - L| \geq \varepsilon. \end{aligned}$$

Now we can provide an equivalent but more useful definition of divergence:

Definition 5. The sequence $\langle a_n \rangle$ is **divergent** if for every possible limit $L \in \mathbb{R}$, there is an $\varepsilon > 0$ such that no matter what $N_\varepsilon \in \mathbb{N}$ we choose, there is an $n > N_\varepsilon$ such that $|a_n - L| \geq \varepsilon$.

Example. Prove that the sequence $\langle a_n \rangle = \langle (-1)^n \rangle$ is divergent.

Preliminary scratchwork: The first quantifier to deal with is $\forall L$. Let's divide L into two cases: $L \geq 0$ and $L < 0$. The second quantifier to deal with is $\exists \varepsilon$. To prove divergence, we only need to show that there is some ε satisfying the criteria. We can choose any error bound we want; let's choose $\varepsilon = \frac{1}{2}$.

In the first case, where $L \geq 0$, we want to guarantee that for any N_ε, we can choose an $n > N_\varepsilon$ such that $|a_n - L| \geq \varepsilon$. Note that

$$|a_n - L| = |(-1)^n - L|.$$

So if n is odd, then

$$|a_n - L| = |-1 - L| = 1 + L$$

(we can evaluate the absolute value since $L \geq 0$). But $1 + L > 1 > \frac{1}{2} = \varepsilon$. No matter how large N_ε is, we can always choose an odd number $n > N_\varepsilon$.

In the second case, where $L < 0$, note that if n is even, then

$$|a_n - L| = |(-1)^n - L| = |1 - L| = 1 - L,$$

(we can remove the absolute value since $L < 0$ and hence $-L > 0$). But $1 - L > 1 > \varepsilon$.

Proof: Let $L \in \mathbb{R}$. Choose $\varepsilon = \frac{1}{2}$. Take any $N_\varepsilon \in \mathbb{N}$.

Case 1: Suppose $L \geq 0$. Choose any odd $n > N_\varepsilon$. Then

$$|a_n - L| = |(-1)^n - L| = |-1 - L| = 1 + L > 1 > \varepsilon.$$

Therefore $\langle a_n \rangle$ does not converge to L.

Case 2: Suppose $L < 0$. Choose any even $n > N_\varepsilon$. Then

$$|a_n - L| = |(-1)^n - L| = |1 - L| = 1 - L > 1 > \varepsilon.$$

Therefore $\langle a_n \rangle$ does not converge to L.

1.3 Diverging to infinity

Definition 6. A sequence $\langle a_n \rangle$ **diverges to infinity** if for any $M \in \mathbb{N}$, there is a natural number N_M such that whenever $n > N$, we have $a_n > M$.

Example. Prove that the sequence $\langle a_n \rangle = \langle 2n^2 \rangle$ diverges to infinity.

Preliminary scratchwork: We want $2n^2 > M$ whenever $n > N_M$. Let's solve for n:

$$2n^2 > M \quad \iff \quad n^2 > \frac{M}{2} \quad \iff \quad n > \sqrt{\frac{M}{2}}.$$

Therefore, if $N_M > \sqrt{\frac{M}{2}}$, then $n > N_M$ implies $n > \sqrt{\frac{M}{2}}$, and hence $2n^2 > M$. Choose $N_M > \left\lceil \sqrt{\frac{M}{2}} \right\rceil$.

Proof: Suppose $M \in \mathbb{N}$. Let $N_M = \left\lceil \sqrt{\frac{M}{2}} \right\rceil$. Then whenever $n > N_M$, we have $n > \sqrt{\frac{M}{2}}$, i.e., $2n^2 > M$. Therefore the sequence $\langle a_n \rangle$ diverges to infinity.

2 Limits of polynomial functions

Definition 7. Let f be a function (specifically, one that takes real numbers as inputs and outputs real numbers), and suppose $a, L \in \mathbb{R}$. The **limit** of f as x approaches a is L, denoted $\lim_{x \to a} f(x) = L$, if for all real numbers $\varepsilon > 0$, there is some real number $\delta_\varepsilon > 0$ such that whenever $x \neq a$ and $|x - a| < \delta_\varepsilon$, we have $|f(x) - L| < \varepsilon$.

Note that

$$\begin{aligned} |x - a| < \delta_\varepsilon &\iff -\delta_\varepsilon < x - a < \delta_\varepsilon \\ &\iff a - \delta_\varepsilon < x < a + \delta_\varepsilon \\ &\iff x \in (a - \delta_\varepsilon, a + \delta_\varepsilon), \end{aligned}$$

so the condition $|x - a| < \delta_\varepsilon$ is equivalent to saying that x is in an interval of radius δ_ε around a, i.e., x is within δ_ε of a.

Proving limits of functions is analogous to proving convergence of sequences. First, for any ε we are given, we get to choose a new δ_ε, and the subscript "ε" emphasizes this dependence. Second, we can consider $|x - a|$ to the error in approximating L by $f(x)$. We might want to bound this error, say by some real number ε (the Greek letter corresponding to "e", for "error"). Ideally, as long as we keep x close enough to a (i.e., within δ_ε of a), then the error $|f(x) - L|$ will remain smaller than our bound ε. In this ideal situation, we say the limit of $f(x)$ is L.

Example. Let $f(x) = 3x - 5$. Prove that $\lim_{x \to 2} f(x) = 1$.

Preliminary scratchwork: For this example, $L = 1$ and $a = 2$. First we examine the error term $|f(x) - L|$:

$$|f(x) - L| = |(3x - 5) - 1| = |3x - 6| = 3|x - 2|.$$

Since we want to bound the error $|f(x) - L|$ by ε, we need $3|x - 2| < \varepsilon$, i.e., $|x - 2| < \frac{\varepsilon}{3}$. Now let's think about $|x - a| = |x - 2|$. If we choose $\delta_\varepsilon = \frac{\varepsilon}{3}$, then $|x - 2| < \delta_\varepsilon$ implies $|x - 2| < \frac{\varepsilon}{3}$, i.e., $|f(x) - L| < \varepsilon$.

Proof: Suppose $\varepsilon > 0$. Choose $\delta_\varepsilon = \frac{\varepsilon}{3}$. Then whenever $|x - 2| < \delta_\varepsilon$, we have $|x - 2| < \frac{\varepsilon}{3}$, i.e., $3|x - 2| < \varepsilon$. So

$$|f(x) - L| = |(3x - 5) - 1| = 3|x - 2| < \varepsilon.$$

Example. Prove that $\lim_{x \to 0} 7x^2 = 0$.

Preliminary scratchwork: For this example, both $L = 0$ and $a = 0$. Again, first we examine the error term:

$$|f(x) - L| = \left|7x^2 - 0\right| = \left|7x^2\right| = 7x^2.$$

We want to make $|f(x) - L| < \varepsilon$, which will occur iff

$$7x^2 < \varepsilon \quad \Longleftrightarrow \quad x^2 < \frac{\varepsilon}{7} \quad \Longleftrightarrow \quad x < \sqrt{\frac{\varepsilon}{7}}.$$

We need to choose δ_ε so that $|x - a| < \delta_\varepsilon$ implies $x < \sqrt{\frac{\varepsilon}{7}}$. Let's examine $|x - a|$:

$$|x - a| = |x - 0| = |x|.$$

If we choose $\delta_\varepsilon = \sqrt{\frac{\varepsilon}{7}}$, then $x < |x| < \sqrt{\frac{\varepsilon}{7}}$, so $7x^2 < \varepsilon$.

Proof: Suppose $\varepsilon > 0$. Let $\delta_\varepsilon = \sqrt{\frac{\varepsilon}{7}}$. Then whenever $|x - a| = |x| < \delta_\varepsilon$, we have $x < |x| < \sqrt{\frac{\varepsilon}{7}}$, i.e., $7x^2 < \varepsilon$. So

$$|f(x) - L| = \left|7x^2 - 0\right| = 7x^2 < \varepsilon.$$

Both of the examples above involve polynomials of low degree, i.e., at most 2. Unfortunately, ε-δ proofs involving higher degree polynomials are a lot more complicated.

Example. Let $p(x) = x^3 - 2x^2 + 3x - 4$. Prove that $\lim_{x \to 1} p(x) = p(1)$.

Preliminary scratchwork: For this example, $L = p(1)$ and $a = 1$. The scratchwork takes four steps:

1. Let's first examine the error term $|p(x) - L| = |p(x) - p(1)|$, which we want to make less than ε. Note $p(1) = 1^3 - 2 \cdot 1^2 + 3 \cdot 1 - 4 = -2$. So

$$|p(x) - L| = |p(x) - p(1)| = |x^3 - 2x^2 + 3x - 4 - (-2)| = |x^3 - 2x^2 + 3x - 2|.$$

2. Consider the polynomial $g(x) = p(x) - p(1)$. Clearly, $g(1) = 0$, i.e., 1 is a root of g. Therefore, by the fundamental theorem of algebra, we can factor $g(x)$ into a product of $(x-1)$ and some other polynomial, say $q(x)$. I.e.,

$$\begin{aligned} g(x) &= (x-1)\cdot q(x) \\ p(x) - p(1) &= (x-1)\cdot q(x) \\ x^3 - 2x^2 + 3x - 2 &= (x-1)\cdot q(x). \end{aligned}$$

   We can calculate $q(x)$ using polynomial long division, getting $q(x) = x^2 - x + 2$. So $p(x) - p(1) = (x-1)(x^2 - x + 2)$. From step 1, we have

$$|p(x) - L| = |p(x) - p(1)| = |x^3 - 2x^2 + 3x - 2| = |x-1||x^2 - x + 2|.$$

3. We know that the smaller δ_ε is, the better an approximation $p(x)$ will be, i.e., the smaller $|p(x) - L|$ will be. So it's safe to choose δ_ε to be smaller than 1. We will make this rigorous when we write up the proof.

   If $|x - a| < \delta_\varepsilon$ and $\delta_\varepsilon \leq 1$, then

$$\begin{aligned} |x - a| < 1 &\iff |x - 1| < 1 \\ &\iff -1 < x - 1 < 1 \\ &\iff 0 < x < 2 \\ &\iff |x| = x < 2. \end{aligned}$$

   From the triangle inequality (recall the triangle inequality is $|a+b| \leq |a| + |b|$), we get

$$\begin{aligned} |q(x)| &= |x^2 + (-x) + 2| \\ &\leq |x^2| + |-x| + |2| \\ &= x^2 + x + 2 \\ &< 2^2 + 2 + 2 \\ &= 8. \end{aligned}$$

4. Now we can finally figure out what δ_ε should be. We want $|p(x) - p(1)| < \varepsilon$. We know that

$$|p(x) - p(1)| = |(x-1)q(x)| = |x-1||q(x)| < 8|x-1|.$$

   So $|p(x) - p(1)| < \varepsilon$ iff $8|x-1| < \varepsilon$ iff $|x-1| < \frac{\varepsilon}{8}$. We will choose $\delta_\varepsilon = \min\left(1, \frac{\varepsilon}{8}\right)$. (Note that $\min(a,b)$ is just the function that takes the smaller of the two numbers a, b. We'll see why we use min when we write up the proof.)

Proof: Take any $\varepsilon > 0$. Let $\delta_\varepsilon = \min\left(1, \frac{\varepsilon}{8}\right)$. We want to show that whenever $x \neq 1$ and $|x-1| < \delta_\varepsilon$, we have $|p(x) - p(1)| < \varepsilon$. So suppose $x \neq 1$ and $|x - 1| < \delta_\varepsilon$. Since $\delta_\varepsilon = \min\left(1, \frac{\varepsilon}{8}\right) \leq 1$, we have $|x-1| < 1$, i.e., $0 < x < 2$. Therefore,

$$|x^2 - x + 2| \leq |x^2| + |-x| + |2| = x^2 + x + 2 < 8.$$

Also, $|x-1| < \delta_\varepsilon \leq \frac{\varepsilon}{8}$. Combining these two inequalities, we get

$$\begin{aligned} |p(x)-p(1)| &= |(x^3-2x^2+3x-4)-(-2)| \\ &= |x^3-2x^2+3x-2| \\ &= |x^2-x+2||x-1| \\ &< 8|x-1| \\ &< 8 \cdot \frac{\varepsilon}{8} \\ &= \varepsilon. \end{aligned}$$

3 Limits of rational functions

Definition 8. A function f is **rational** if it is a quotient of two polynomial functions, i.e., $f(x) = \frac{p(x)}{q(x)}$ where p, q are polynomials.

Let's recall some facts about rational functions from calculus. The domain of a rational function $f(x) = \frac{p(x)}{q(x)}$ is whenever the denominator is nonzero, i.e., $\{x \in \mathbb{R} : q(x) \neq 0\}$. If the denominator $q(x)$ is zero at $x = a$, then two kinds of discontinuities can occur:

- There can be a "removable discontinuity" at $x = a$, which appears visually as a hole in the graph. This occurs when $(x-a)$ is a factor of both $q(x)$ and $p(x)$. Then in the limit, this factor can be cancelled, thus "removing" the discontinuity.

  For example, let $f(x) = \frac{x^2+1}{x-1}$. Then f has a discontinuity $x = 1$. Note that $f(x) = \frac{(x+1)(x-1)}{x-1}$, but if we cancel the factor $(x-1)$ in the numerator and denominator, then we change the domain and thus change the function. I.e., *we cannot cancel* $(x-1)$ *while retaining the original function* f. Let's call the new function we get after cancelling g, i.e., $g(x) = x+1$. At every point b other than 1, we have $f(b) = \frac{(b+1)(b-1)}{b-1} = b+1 = g(b)$ (we can cancel here because $b \neq 1$, and we are just evaluating the number $f(b)$). So the graphs of f and g look the same for all $x \neq 1$. When $x = 1$, we have $g(x) = 2$ but $f(x)$ is not defined. So the graph of g is a continuous straight line, and the graph of f is that same line but with a hole at $x = 1$.

- There can be a vertical asymptote at $x = a$. This occurs when $(x-a)$ is a factor of $q(x)$ but not a factor of $p(x)$. For example $f_1(x) = \frac{1}{x}$ and $f_2(x) = \frac{1}{x^2}$ have vertical asymptotes at $x = 0$.

Note that for any a in the domain of a rational function $f(x) = \frac{p(x)}{q(x)}$ (i.e., for any a such that $q(a) \neq 0$), f is continuous at a. Let's explore each of these three situations using ε-δ proofs.

3.1 Removable discontinuities

Example. Prove that $\lim\limits_{x \to 1} \frac{x^2+1}{x-1} = 2$.

Preliminary scratchwork: Let $f(x) = \frac{x^2+1}{x-1}$, $L = 2$, and $a = 1$. As usual, we first examine the error:

$$|f(x) - L| = \left|\frac{x^2+1}{x-1} - 2\right| = \left|\frac{(x^2+1)-(2x-2)}{x-1}\right| = \left|\frac{x^2-2x+1}{x-1}\right| = \left|\frac{(x-1)^2}{x-1}\right|.$$

Recall that in the definition of a limit, we want $|f(x)-L| < \varepsilon$ whenever we have both $x \neq a$ and $|x-a| < \delta_\varepsilon$. Since $x \neq a$, i.e., $x \neq 1$, it's okay to cancel in the above fraction:

$$|f(x) - L| = \left|\frac{(x-1)^2}{x-1}\right| = |x-1|.$$

So $|f(x) - L| < \varepsilon$ whenever $|x-a| < \varepsilon$. We can just choose δ_ε as ε.

Proof: Let $\varepsilon > 0$. Choose $\delta_\varepsilon = \varepsilon$. Then whenever $x \neq 1$ and $|x-a| = |x-1| < \delta_\varepsilon = \varepsilon$, we have

$$|f(x) - L| = \left|\frac{x^2+1}{x-1} - 2\right| = \left|\frac{(x-1)^2}{x-1}\right| = |x-1| < \varepsilon.$$

3.2 Vertical asymptotes

We need an ε-δ style definition analogous to definition 6 about sequences diverging to infinity.

Definition 9. A function f **diverges to infinity** as x approaches $a \in \mathbb{R}$, denoted $\lim_{x\to a} f(x) = \infty$, if for all real numbers $M > 0$, there is some real number $\delta_M > 0$ such that whenever $x \neq a$ and $|x-a| < \delta_M$, we have $f(x) > M$.

Note that this definition states that on *both* the left and right sides of the vertical asymptote, $f(x)$ goes to infinity, i.e., $\lim_{x\to a^-} f(x) = \infty$ and $\lim_{x\to a^+} f(x) = \infty$. This does not encapsulate the situation where on one side $f(x)$ goes to $-\infty$ and on the other side $f(x)$ goes to $+\infty$, e.g., when $f(x) = \frac{1}{x}$. To analyze this situation, one needs a rigorous definition of $\lim_{x\to a-} f(x) = -\infty$ and $\lim_{x\to a+} f(x) = \infty$ (which you can get by tweaking definition 9, e.g., replacing "$|x-a| < \delta_M$" with "$a < x < a+\delta_M$" gives you the definition of $\lim_{x\to a+} f(x) = \infty$). In these notes we will only consider the situation where $\lim_{x\to a^-} f(x) = \infty = \lim_{x\to a^+} f(x)$, since the other situations are analogous.

Example. Prove that $\lim_{x\to 0} \frac{1}{x^2} = \infty$.

Preliminary scratchwork: Let $f(x) = \frac{1}{x^2}$ and $a = 0$. Then

$$f(x) > M \iff \frac{1}{x^2} > M \iff x < \frac{1}{\sqrt{M}}.$$

We want $f(x) > M$ whenever $|x - 0| < \delta_M$. Let's choose $\delta_M = \frac{1}{\sqrt{M}}$.

Proof: Let $M > 0$. Choose $\delta_M = \frac{1}{\sqrt{M}}$. Then $|x-0| < \delta_M$ implies $x < |x| < \delta_M = \frac{1}{\sqrt{M}}$, which implies $\frac{1}{x^2} > M$. So whenever $|x-a| < \delta_M$, we have $f(x) = \frac{1}{x^2} > M$.

3.3 Limits of rational functions at points of continuity

If $f(x) = \frac{p(x)}{q(x)}$ and $q(a) \neq 0$, then $f(x)$ is continous at a, which we can prove with a usual ε-δ argument.

Example. Prove $\lim_{x \to 5} \frac{1}{x} = \frac{1}{5}$.

Preliminary scratchwork: Let $f(x) = \frac{1}{x}$, $L = \frac{1}{5}$, and $a = 5$. First we examine the error:

$$|f(x) - L| = \left|\frac{1}{x} - \frac{1}{5}\right| = \left|\frac{5-x}{5x}\right| = \frac{|5-x|}{5|x|}.$$

We want to bound $|f(x) - L| < \varepsilon$ by bounding $|x - a| < \delta_\varepsilon$. Note $|x - a| = |x - 5| = |5 - x|$. Although $|5 - x|$ appears in our expression above for $|f(x) - L|$, we also have $|x|$ in the denominator. We need to set a rough bound on $|x|$, as we did when writing ε-δ proofs for limits of higher degree polynomials in section 2.

Let's choose $\delta_\varepsilon \leq 1$; we'll make this rigorous when we write up the proof. This choice is safe because an interval of radius 1 centered at $x = 5$ avoids the discontinuity at $x = 0$. Since $\delta_\varepsilon \leq 1$, we have

$$\begin{aligned} |x-5| < \delta_\varepsilon &\iff |x-5| < 1 \\ &\iff -1 < x - 5 < 1 \\ &\iff 4 < x < 6 \\ &\iff \frac{1}{4} > \frac{1}{x} > \frac{1}{6} \\ &\implies \frac{1}{|x|} < \frac{1}{4} \end{aligned}$$

So if $|x - 5| < \delta_\varepsilon < 1$, then

$$|f(x) - L| = \frac{|5-x|}{5|x|} = \frac{|5-x|}{5} \cdot \frac{1}{|x|} < \frac{|5-x|}{5} \cdot \frac{1}{4} = \frac{|5-x|}{20}.$$

We want $|f(x) - L| < \varepsilon$, i.e., $|x - 5| < 20\varepsilon$, whenever $|x - 5| < \delta_\varepsilon$. So we need $\delta_\varepsilon \leq 20\varepsilon$. We already chose $\delta_\varepsilon \leq 1$, so let's combine these two conditions by defining $\delta_\varepsilon = \min(1, 20\varepsilon)$.

Proof: Take any $\varepsilon > 0$. Choose $\delta_\varepsilon = \min(1, 20\varepsilon)$. We want to show that whenever $|x-5| < \delta_\varepsilon$, we have $|f(x) - L| < \varepsilon$. So suppose $|x - 5| < \delta_\varepsilon$. In particular, we have $|x - 5| < 20\varepsilon$ and also $|x - 5| < 1$, so

$$\begin{aligned} -1 < x - 5 < 1 &\iff 4 < x < 6 \\ &\iff \frac{1}{4} > \frac{1}{x} > \frac{1}{6} \\ &\implies \frac{1}{|x|} < \frac{1}{4}. \end{aligned}$$

Therefore,

$$|f(x) - L| = \left|\frac{1}{x} - \frac{1}{5}\right| = \frac{|5-x|}{5|x|} < \frac{|5-x|}{5} \cdot \frac{1}{4} = \frac{|5-x|}{20} < \frac{20\varepsilon}{20} = \varepsilon.$$