Math 156: Review problems for Exam 1
All but one of the problems below are odd-numbered problems from the textbook. The solutions are in the back of the book.

Sets

1.4.7 Write out $\mathcal{P}(\{a,b\}) \times \mathcal{P}(0,1)$

1.5.3(g) Write out $\mathcal{P}(\{0,1\}) - \mathcal{P}(\{1,2\})$.

1.8.5 Find the sets $\bigcup_{i\in\mathbb{N}}[i,i+1]$ and $\bigcap_{i\in\mathbb{N}}[i,i+1]$.

Symbolic logic

2.6.9 Using a truth table, decide if $P \wedge Q$ and $\sim(\sim P \vee \sim Q)$ are logically equivalent.

*** (not in book, solution below)** Let P be the statement, "Given any $x \in \mathbb{R}$, there exists an element $y \in \mathbb{R}$ such that $xy = 1$".
(a) Is P true or false? Explain your reasoning (no need to formally prove it if it's true).
(b) Translate P in symbolic logic, then write the negation $\sim P$ in symbolic logic (make sure all quantifiers are properly negated in your final answer).

2.9.7 Translate the following statement into symbolic logic: "There exists a real number a for which $a + x = x$ for every real number x."

2.10.3 Negate the following sentence: "For every prime number p, there is another prime number q with $q > p$."

Direct proof

4.7 Prove: Suppose $a, b \in \mathbb{Z}$. If $a \mid b$, then $a^2 \mid b^2$.

4.19 Prove: Suppose $a, b, c \in \mathbb{Z}$. If $a^2 \mid b$ and $b^3 \mid c$, then $a^6 \mid c$.

4.15 Prove: If $n \in \mathbb{Z}$, then $n^2 + 3n + 4$ is even. (Try cases.)

Induction

10.3 Prove: For every integer $n \in \mathbb{N}$, it follows that $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$.

10.5 Prove: If $n \in \mathbb{N}$, then $2^1 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$.

Proof by contrapositive and contradiction

5.7 Prove by contrapositive: If both ab and $a + b$ are even, then both a and b are even.

5.11 Prove by contrapositive: Suppose $x, y \in \mathbb{Z}$. If $x^2(y+3)$ is even, then x is even or y is odd.

6.9 Prove by contradiction: Suppose $a, b \in \mathbb{R}$. If a is rational and ab is irrational, then b is irrational.

6.15 Prove by contradiction: If $b \in \mathbb{Z}$ and $b \nmid k$ for every $k \in \mathbb{N}$, then $b = 0$.

Solution to problem * above: Let P be the statement, "Given any $x \in \mathbb{R}$, there exists an element $y \in \mathbb{R}$ such that $xy = 1$".

(a) Is P true or false? Explain your reasoning (no need to formally prove it if it's true).

P is false. Here's a counterexample: if $x = 0$, then for all $y \in \mathbb{R}$, we have $xy = 0 \cdot y = 0$.

(b) Translate P in symbolic logic, then write the negation $\sim P$ in symbolic logic (make sure all quantifiers are properly negated in your final answer).

$$P \text{ is } \forall x \in \mathbb{R}\ \ \exists y \in \mathbb{R}\ \ xy = 1$$

$$\begin{aligned} \sim P \text{ is } &\sim(\forall x \in \mathbb{R}\ \ \exists y \in \mathbb{R}\ \ xy = 1) \\ &\exists x \in \mathbb{R}\ \ \sim(\exists y \in \mathbb{R}\ \ xy = 1) \\ &\exists x \in \mathbb{R}\ \ \forall y \in \mathbb{R}\ \ \sim(xy = 1) \\ &\exists x \in \mathbb{R}\ \ \forall y \in \mathbb{R}\ \ xy \neq 1 \end{aligned}$$