Math 156: Exam 2 – *Solutions*

1. (10 points)

Suppose A, B are sets. Prove $A \cap B = A - (A - B)$, by first showing $A \cap B \subseteq A - (A - B)$ and then showing $A \cap B \supseteq A - (A - B)$. (Venn diagrams do not suffice for a proof.)

Solution: Let $x \in A \cap B$, i.e., $x \in A$ and $x \in B$. Since $x \in B$, we have $x \notin A - B$. So x is in A but not in $A - B$, i.e., $x \in A - (A - B)$. Hence $A \cap B \subseteq A - (A - B)$.

Conversely, suppose $x \in A - (A - B)$. Then

$$\begin{aligned}
(x \in A) \wedge \sim(x \in A - B) &\iff (x \in A) \wedge \sim(x \in A \wedge x \notin B) \\
&\iff (x \in A) \wedge (x \notin A \vee x \in B) \\
&\iff (x \in A \wedge x \notin A) \vee (x \in A \wedge x \in B) \\
&\iff (\text{false}) \vee (x \in A \wedge x \in B) \\
&\iff x \in A \wedge x \in B \\
&\iff x \in A \cap B.
\end{aligned}$$

Hence $A - (A - B) \subseteq A \cap B$.

(Note that the series of "$\iff$" equivalences in the second paragraph is actually enough to show that $A - (A - B) = A \cap B$.)

2. (10 points)

Let $A = \{1, 2, 3, 4\}$. Give an example of

- a relation R_1 on A that is reflexive and symmetric but not transitive,
- a relation R_2 on A that is reflexive and transitive but not symmetric, and
- a relation R_3 on A that is symmetric and transitive but not reflexive.

Solution: Here's one possible list of examples:

$$\begin{aligned}
R_1 &= \{(1,1), (2,2), (3,3), (4,4), (1,2), (2,1), (2,3), (3,2)\} \\
R_2 &= \{(1,1), (2,2), (3,3), (4,4), (1,2), (2,3), (1,3)\} \\
R_3 &= \{(1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (3,2), (1,3), (3,1)\}.
\end{aligned}$$

3. (10 points)

Let $W = \{0, 1, 2, 3, \ldots\} = \{0\} \cup \mathbb{N}$. Define a relation R on $W \times W$ by $(x_1, x_2)\, R\, (y_1, y_2)$ if and only if $x_1 + y_2 = y_1 + x_2$. Prove R is an equivalence relation.

Solution: To show R is reflexive, take any $x_1, x_2 \in W$. Then $x_1 + x_2 = x_1 + x_2$, so by definition of R, we have $(x_1, x_2)R(x_1, x_2)$.

To show R is symmetric, suppose $(x_1, x_2)R(y_1, y_2)$. Then $x_1 + y_2 = y_1 + x_2$. Swapping the left-hand side with the right-hand side, we get $y_1 + x_2 = x_1 + y_2$. So $(y_1, y_2)R(x_1, x_2)$.

To show R is transitive, suppose $(x_1, x_2)R(y_1, y_2)$ and $(y_1, y_2)R(z_1, z_2)$. The former gives us $x_1 + y_2 = y_1 + x_2$, i.e., $x_1 - x_2 = y_1 - y_2$. The latter gives us $y_1 + z_2 = z_1 + y_2$, i.e., $y_1 - y_2 = z_1 - z_2$. Combining these equations, we get $x_1 - x_2 = z_1 - z_2$, or equivalently, $x_1 + z_2 = z_1 + x_2$. Therefore $(x_1, x_2)R(z_1, z_2)$.

4. (10 points)

Let W and R be as defined in problem 3. Define C to be the set of equivalence classes of $W \times W$ under the equivalence relation R, i.e., $C = \Big\{\big[(a, b)\big] : a, b \in W\Big\}$. Define the function $f : C \to \mathbb{Z}$ by $\big[(m, n)\big] \mapsto m - n$. Prove that f is injective and surjective.

Solution: We will show f is injective by contrapositive. Suppose $f(\big[(m_1, n_1)\big]) = f(\big[(m_2, n_2)\big])$. Then $m_1 - n_1 = m_2 - n_2$. I.e., $m_1 + n_2 = m_2 + n_1$, so $(m_1, n_1)R(m_2, n_2)$. Then they're in the same equivalence class, i.e., $\big[(m_1, n_1)\big] = \big[(m_2, n_2)\big]$. So f is injective.

To show f is surjective, take any $k \in \mathbb{Z}$. If $k \geq 0$, then $k \in W$, so $\big[(k, 0)\big] \in C$. Then $f(\big[(k, 0)\big]) = k - 0 = k$. If $k < 0$, then $-k \in W$, so $\big[(0, -k)\big] \in C$. Then $f(\big[(0, -k)\big]) = 0 - (-k) = k$. In either case, we have found an element in C that is sent to k.

5. (10 points)

Let W be as defined in problem 3. Define a function $f : W \times W \to \mathbb{Z}$ by $(a, b) \mapsto (-1)^a b$. Prove or disprove whether f is injective. Also prove or disprove whether f is surjective.

Solution: We will show f is not injective by counterexample. Namely, $f(2, 5) = (-1)^2 \cdot 5 = 5$ and $f(4, 5) = (-1)^4 \cdot 5 = 5$, but $(2, 5) \neq (4, 5)$.

To show f is surjective, suppose $k \in \mathbb{Z}$. If $k = 0$, then for any $a \in W$, we have $f(a, 0) = (-1)^a \cdot 0 = 0$. If $k > 0$, then $(0, k) \in W \times W$ and $f(0, k) = (-1)^0 k = k$. If $k < 0$, then $(1, -k) \in W \times W$ and $f(1, -k) = (-1)^1(-k) = k$.

6. (10 points)

Suppose $f : A \to B$ and $Y_1, Y_2 \subseteq B$. Prove

$$f^{-1}(Y_1 \cap Y_2) = f^{-1}(Y_1) \cap f^{-1}(Y_2)$$

by first showing $f^{-1}(Y_1 \cap Y_2) \subseteq f^{-1}(Y_1) \cap f^{-1}(Y_2)$ and then showing $f^{-1}(Y_1 \cap Y_2) \supseteq f^{-1}(Y_1) \cap f^{-1}(Y_2)$.

Solution: To show ($\subseteq$), suppose $a \in f^{-1}(Y_1 \cap Y_2)$. Then by the definition of a preimage, we have $f(a) \in Y_1 \cap Y_2$. So $f(a) \in Y_1$ and $f(a) \in Y_2$, i.e., $a \in f^{-1}(Y_1)$ and $a \in f^{-1}(Y_2)$. So $a \in f^{-1}(Y_1) \cap f^{-1}(Y_2)$.

To show ($\supseteq$), suppose $a \in f^{-1}(Y_1) \cap f^{-1}(Y_2)$. Then $a \in f^{-1}(Y_1)$ and $a \in f^{-1}(Y_2)$, i.e., $f(a) \in Y_1$ and $f(a) \in Y_2$. So $f(a) \in Y_1 \cap Y_2$. Therefore $a \in f^{-1}(Y_1 \cap Y_2)$.