Math 156: Review problems for the final exam

Proving "if and only if" statements

7.3 Given an integer a, $a^3 + a^2 + a$ is even if and only if a is even.

7.15 Suppose $a, b \in \mathbb{Z}$. Prove that $a + b$ is even if and only if a and b have the same parity.

Proofs involving sets

8.9 If A, B, C are sets, then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

8.13 If A, B, C are sets, then $A - (B \cup C) = (A - B) \cap (A - C)$.

8.21 Suppose A and B are sets. Prove $A \subseteq B$ if and only if $A - B = \varnothing$.

8.31 Suppose $B \neq \varnothing$ and $A \times B \subseteq B \times C$. Prove $A \subseteq C$.

Relations

11.2.9. (modified) Define a relation on $\mathbb{Z}$ by declaring xRy if and only if x and y have the same parity. Is R reflexive? symmetric? transitive? For each property, either prove R has the property or give a counterexample that lacks the property.

11.2.13. Consider the relation $R = \{(x, y) \in \mathbb{R} : x - y \in \mathbb{Z}\}$ on the set $\mathbb{R}$. Prove that this relation is reflexive, symmetric, and transitive.

11.3.7. Define a relation R on $\mathbb{Z}$ as xRy if and only if $3x - 5y$ is even. Prove R is an equivalence relation.

11.3.9. Define a relation R on $\mathbb{Z}$ as xRy if and only if $4 \mid (x+3y)$. Prove R is an equivalence relation.

Functions

12.1.9. Consider the set $f = \{(x^2, x) : x \in \mathbb{R}\}$. Is this a function from $\mathbb{R}$ to $\mathbb{R}$? Explain.

12.2.1. Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$. Give an example of a function $f : A \to B$ that is neither injective nor surjective.

12.2.5. A function $f : \mathbb{Z} \to \mathbb{Z}$ is defined as $f(n) = 2n + 1$. Is this function injective? surjective? For each property, either prove f has the property or give a counterexample.

12.2.7. A function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is defined as $f(m, n) = 2n - 4m$. Is this function injective? surjective? For each property, either prove f has the property or give a counterexample.

12.2.11. Consider the function $\theta : \{0,1\} \times \mathbb{N} \to \mathbb{Z}$ defined as $\theta(a,b) = (-1)^a b$. Is θ injective? surjective? For each property, either prove θ has the property or give a counterexample.

12.6.5. Let $f : A \to B$ and $X \subseteq A$. Prove that $X \subseteq f^{-1}(f(X))$.

12.6.11. Given a function $f : A \to B$ and subsets $Y, Z \subseteq B$, prove $f^{-1}(Y \cup Z) = f^{-1}(Y) \cup f^{-1}(Z)$.

(*) Suppose $f : A \to B$ and $g : B \to C$, and $g \circ f$ is a bijection from A to C (recall that $g \circ f(a) = g(f(a))$). Show by way of contradiction that f is injective and g is surjective. (Solution below.)

Cardinality

Note: One of the following three problems will be on the final exam. Their solutions can be found in your class notes or in sections 14.1 and 14.2 of the textbook.

1. Describe a bijection that shows $|\mathbb{N}| = |\mathbb{Z}|$ (it doesn't have to be rigorous).

2. Describe a bijection that shows $|\mathbb{N}| = |\mathbb{Q}|$ (it doesn't have to be rigorous).

3. Prove that $|\mathbb{N}| \neq |\mathbb{R}|$.

Some solutions

(*) Suppose $f : A \to B$ and $g : B \to C$, and $g \circ f$ is a bijection from A to C (recall that $(g \circ f)(a) = g(f(a))$). Show by way of contradiction that f is injective and g is surjective.

First we'll prove f is injective. Suppose by way of contradiction that it isn't. Then there are $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$. Then $g(f(a_1)) = g(f(a_2))$, i.e., $(g \circ f)(a_1) = (g \circ f)(a_2)$. But then $g \circ f$ is not injective, contradicting the assumption that it's bijective.

Now let's prove g is surjective. Suppose by way of contradiction that it isn't. Then there is some $c \in C$ such that for all $b \in B$, $g(b) \neq c$. So for any $a \in A$, we have $f(a) \in B$ and hence $g(f(a)) \neq c$. I.e., $g \circ f$ is not surjective, contradicting the assumption that it's bijective.